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FAST TRACK COMMUNICATION

Multi-operator brackets acting thrice

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Online at stacks.iop.org/JPhysA/42/462001**Abstract**

We generalize an identity first found by Bremner for Nambu 3-brackets. For odd N -brackets built from associative operator products, we show that

$$\begin{aligned} & [[A[B_1 \cdots B_N]B_{N+1} \cdots B_{2N-2}]B_{2N-1} \cdots B_{3N-3}] \\ &= [[AB_1 \cdots B_{N-1}][B_N \cdots B_{2N-1}]B_{2N} \cdots B_{3N-3}] \end{aligned}$$

for any fixed A , when totally antisymmetrized over all the B s.

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1. Introduction

Nambu introduced a multilinear operator bracket in the context of a novel formulation of mechanics [13]. His N -bracket is defined by

$$[A_1 A_2 \cdots A_N] = \sum_{\sigma \in S_N} \text{sgn}(\sigma) A_{\sigma_1} \cdots A_{\sigma_N}, \quad (1)$$

where the sum is over all $N!$ permutations of the operators. For example, $[ABC] = ABC - ACB + BCA - BAC + CAB - CBA$. The operator product is assumed to be associative. To avoid ambiguities when some of the entries within a bracket are themselves products, commas are often used to separate the entries. Parentheses also suffice in such cases. For example, $[AD, B, C] = [(AD)BC] = AD BC - ADC B + BC AD - BADC + CAD B - CBAD$.

The same construction independently appeared in the mathematical literature [10, 11]. The theory of such multi-operator products, as well as their ‘classical limits’ in terms of multivariable Jacobians, has been studied extensively [1–4, 6–9, 12, 14–18].

From an algebraic point of view, it is natural to seek the analogue of the Jacobi identity for N -brackets. For the case of even N -brackets, the obvious generalization where one N -bracket acts on another leads to a true identity *if* all entries are totally antisymmetrized (see (5) below). But for odd N -brackets this procedure does not work [1, 4]—the total antisymmetrization over all entries of one odd N -bracket acting on another does not vanish, but rather yields a higher order $(2N - 1)$ -bracket.

Nevertheless, an interesting generalization of the Jacobi identity was discovered by Bremner for 3-brackets acting three times [1]. He showed

$$[[A[bcd]e]fg] = [[Abc][def]g], \tag{2}$$

where A is fixed, but it is implicitly understood that lowercase entries are totally antisymmetrized by summing over all $6!$ signed permutations of them. The point of this short communication is to show that Bremner's identity generalizes to all odd N -brackets.

Before discussing the general case, we anticipate and indicate a proof for the case of 3-brackets. The Bremner identity can be proved through a resolution of both left- and right-hand sides as a series of canonically ordered words. By direct calculation we find

$$[[A[bcd]e]fg] = 24Abcdefg - 36bAcdefg + 36bcAdefg - 24bcdAefg + 36bcdeAfg - 36bcdefAg + 24bcdefgA, \tag{3}$$

where all lowercase entries are implicitly totally antisymmetrized. Precisely the same expansion holds for $[[Abc][def]g]$, again by direct calculation. Hence, the identity is established.

That is to say, both $[[A[bcd]e]fg]$ and $[[Abc][def]g]$ can be rendered as a 7-bracket plus another 3-bracket containing 3-brackets, when antisymmetrized over lowercase entries:

$$[[A[bcd]e]fg] = \frac{1}{20}[Abcdefg] - \frac{1}{6}[A[bcd][efg]] = [[Abc][def]g]. \tag{4}$$

Thus, the Bremner identity amounts to the combinatorial statement, as written, that there are two distinct ways to present a 7-bracket in terms of nested 3-brackets.

2. Results for any N

As known, and previously mentioned, even brackets need only act twice to yield an identity. Namely [4, 9]

$$[B_1 \cdots B_{N-1}[B_N \cdots B_{2N-1}]] = 0 \quad \text{for } N \text{ even.} \tag{5}$$

Total antisymmetrization of all the B s is understood¹. When $N = 2$ this is the familiar Jacobi identity. The proof is by direct calculation and follows as a consequence of associativity.

However, for odd N , $[B_1 \cdots B_{N-1}[B_N \cdots B_{2N-1}]] \neq 0$, but instead produces the $(2N - 1)$ -bracket $[B_1 \cdots B_{2N-1}]$ upon total antisymmetrization [3, 4]. Apparently, the simplest identity obeyed by odd brackets of only one type, that does *not* introduce higher order brackets, requires that they act at least three times. For any odd $N = 2L + 1$, a valid relation is the immediate generalization of that found by Bremner for the case of 3-brackets. To show this, we present two easily established lemmata, followed by our main theorem and its proof. Firstly,

Lemma 1.

$$[AB_1 \cdots B_J] = J! \sum_{j=0}^J (-1)^j B_1 \cdots B_j AB_{j+1} \cdots B_J. \tag{6}$$

Total antisymmetrization of the B s is understood. Here we have also used the convention that an empty product equals 1. Explicitly, $B_1 \cdots B_0 = 1 = B_{J+1} \cdots B_J$, so that the first and last terms in the sum are $AB_1 \cdots B_J$ and $(-1)^J B_1 \cdots B_J A$, respectively. It is a simple exercise to use this first lemma to prove (5). Similarly,

¹ We will not discuss *all* possible symmetrizations of the bracket entries, but just particular choices that work to give identities for general N . For a thorough study of other symmetrizations, particularly in the 3-bracket case, see [1], as well as the more recent work of Nuyts *et al* [5].

Lemma 2.

$$\begin{aligned}
 [AB_1 \cdots B_J \mathcal{Z}] &= J! \sum_{j=0}^J (-1)^j \sum_{k=0}^{J-j} (-1)^k B_1 \cdots B_k \mathcal{A} B_{k+1} \cdots B_{J-j} \mathcal{Z} B_{J-j+1} \cdots B_J \\
 &\quad - J! \sum_{j=0}^J (-1)^j \sum_{k=0}^{J-j} (-1)^k B_1 \cdots B_k \mathcal{Z} B_{k+1} \cdots B_{J-j} \mathcal{A} B_{J-j+1} \cdots B_J. \tag{7}
 \end{aligned}$$

Finally, it is rather tedious but fairly straightforward to use both lemmata to prove the following.

Theorem 1. For associative products, with implicit total antisymmetrization of the Bs,

$$\begin{aligned}
 [[A[B_1 \cdots B_{2L+1}]B_{2L+2} \cdots B_{4L}]B_{4L+1} \cdots B_{6L}] \\
 = [[AB_1 \cdots B_{2L}][B_{2L+1} \cdots B_{4L+1}]B_{4L+2} \cdots B_{6L}]. \tag{8}
 \end{aligned}$$

Proof. The result (8) follows from resolving the left- and right-hand sides into sums of canonically ordered words, as illustrated above for the case of 3-brackets. We have

$$[[AB_1 \cdots B_{2L}][B_{2L+1} \cdots B_{4L+1}]B_{4L+2} \cdots B_{6L}] = \sum_{n=0}^{6L} (-1)^n m_n^{(1)} B_1 \cdots B_n \mathcal{A} B_{n+1} \cdots B_{6L}, \tag{9}$$

$$[[A[B_1 \cdots B_{2L+1}]B_{2L+2} \cdots B_{4L}]B_{4L+1} \cdots B_{6L}] = \sum_{n=0}^{6L} (-1)^n m_n^{(2)} B_1 \cdots B_n \mathcal{A} B_{n+1} \cdots B_{6L}.$$

All the coefficients $m_n^{(1,2)}$ in these two resolutions are manifestly positive integers. The theorem is established by showing that $m_n^{(1)} = m_n^{(2)}$ for all n .

By direct calculation, through the use of the two lemmata, we find

$$m_n^{(1)} = m_n^{(2)} = (2L + 1)!(2L)!(2L - 1)! \times c_n, \tag{10}$$

$$c_n = \begin{cases} (n + 1)(4L - n)/2 & \text{for } 0 \leq n \leq 2L \\ 10L^2 - 6Ln + L + n^2 & \text{for } 2L + 1 \leq n \leq 3L \\ c_{6L-n} & \text{for } 3L + 1 \leq n \leq 6L \end{cases} \tag{11}$$

The determination of the m_n s is just a matter of enumerating the ways to obtain a particular intercalation of A among the Bs.

Consider in more detail some of the calculations involved. As a first step, with the implicit antisymmetrization², the internal brackets $[B_1 \cdots B_{2L+1}]$ or $[B_{2L+1} \cdots B_{4L+1}]$ may be supplanted by products: $[B_1 \cdots B_{2L+1}] = (2L + 1)!(B_1 \cdots B_{2L+1})$ or $[B_{2L+1} \cdots B_{4L+1}] = (2L + 1)!(B_{2L+1} \cdots B_{4L+1})$. Then we may write, on the one hand,

$$\begin{aligned}
 [[AB_1 \cdots B_{2L}][B_{2L+1} \cdots B_{4L+1}]B_{4L+2} \cdots B_{6L}] \\
 = -(2L + 1)! [[AB_1 \cdots B_{2L}]B_{4L+2} \cdots B_{6L}(B_{2L+1} \cdots B_{4L+1})] \tag{12}
 \end{aligned}$$

² To avoid any misunderstanding, by *implicit total antisymmetrization* of the Bs in the expression $[[A[B_1 \cdots B_{2L+1}]B_{2L+2} \cdots B_{4L}]B_{4L+1} \cdots B_{6L}]$, we mean

$$\sum_{\sigma \in S_{6N}} \text{sgn}(\sigma) [[A[B_{\sigma_1} \cdots B_{\sigma_{2L+1}}]B_{\sigma_{2L+2}} \cdots B_{\sigma_{4L}}]B_{\sigma_{4L+1}} \cdots B_{\sigma_{6L}}],$$

where the sum is over all $(6N)!$ permutations of the indices, $1, \dots, 6N$. Similar meanings apply to the other implicitly antisymmetrized expressions in the communication.

In this expression, we may now rename indices, bearing in mind the antisymmetrization:

$$[[AB_1 \cdots B_{2L}][B_{2L+1} \cdots B_{4L+1}]B_{4L+2} \cdots B_{6L}] = (2L + 1)! [[AB_{2L} \cdots B_{4L-1}]B_1 \cdots B_{2L-1}(B_{4L} \cdots B_{6L})]. \tag{13}$$

Next, we apply lemma 2 for $J = 2L - 1$, and identify $[AB_{2L} \cdots B_{4L-1}]$ with \mathcal{A} , and $(B_{4L} \cdots B_{6L})$ with \mathcal{Z} :

$$[AB_1 \cdots B_{2L-1}\mathcal{Z}] = (2L - 1)! \sum_{j=0}^{2L-1} (-1)^j \times \sum_{k=0}^{2L-1-j} (-1)^k B_1 \cdots B_k \mathcal{A} B_{k+1} \cdots B_{2L-1-j} \mathcal{Z} B_{2L-j} \cdots B_{2L-1} \tag{14a}$$

$$-(2L - 1)! \sum_{j=0}^{2L-1} (-1)^j \sum_{k=0}^{2L-1-j} (-1)^k B_1 \cdots B_k \mathcal{Z} B_{k+1} \cdots B_{2L-1-j} \mathcal{A} B_{2L-j} \cdots B_{2L-1}. \tag{14b}$$

To continue, consider first the coefficients $m_n^{(1)}$ where $n \leq 2L$.

For the determination of $m_{n \leq 2L}^{(1)}$, since \mathcal{Z} consists of $(2L + 1)$ Bs, it *must* be placed to the *right* of \mathcal{A} in the application of lemma 2. Otherwise there would be too many Bs to the left of \mathcal{A} . Thus, for $m_{n \leq 2L}^{(1)}$ we need keep only the first line in the last relation, (14a). To place a total of n Bs to the left of the \mathcal{A} contained in $\mathcal{A} = [AB_{2L} \cdots B_{4L-1}]$, with k Bs already to the left as in (14a), we then need only the terms in \mathcal{A} with an additional $(n - k)$ Bs to the left of \mathcal{A} . That is to say, from lemma 1, with $J = 2L$ and all B indices shifted up by $2L - 1$,

$$\mathcal{A} = [AB_{2L} \cdots B_{4L-1}] = (2L)! \sum_{l=0}^{2L} (-1)^l B_{2L} \cdots B_{l+2L-1} \mathcal{A} B_{l+2L} \cdots B_{4L-1}, \tag{15}$$

and from this we need only the term with $l = n - k$. The net result for $m_{n \leq 2L}^{(1)}$ is

$$m_{n \leq 2L}^{(1)} = (2L + 1)!(2L)!(2L - 1)! \times c_{n \leq 2L}, \tag{16}$$

$$c_{n \leq 2L} = \sum_{j=0}^{2L-1} \sum_{k=0}^{2L-1-j} \sum_{l=0}^{2L} \delta_{l, n-k} \Big|_{n \leq 2L} = \sum_{j=0}^{2L-1} \sum_{k=0}^{\min(n, 2L-1-j)} 1 = \frac{(n + 1)(4L - n)}{2}. \tag{17}$$

On the other hand, with similar steps, we have

$$[[A[B_1 \cdots B_{2L+1}]B_{2L+2} \cdots B_{4L}]B_{4L+1} \cdots B_{6L}] = (2L + 1)! [[AB_1 \cdots B_{2L-1}(B_{2L} \cdots B_{4L})]B_{4L+1} \cdots B_{6L}]. \tag{18}$$

We again apply lemma 2 for $J = 2L - 1$, but to $[AB_1 \cdots B_{2L-1}(B_{2L} \cdots B_{4L})]$, so now we identify \mathcal{A} with \mathcal{A} , and $(B_{2L} \cdots B_{4L})$ with \mathcal{Z} . As before, consider first only $m_n^{(2)}$ coefficients where $n \leq 2L$. For the determination of $m_{n \leq 2L}^{(2)}$, \mathcal{Z} *must* once again be placed to the *right* of \mathcal{A} , so we need keep only the line (14a). We pick up an additional $(n - k)$ Bs by applying again lemma 1, only this time to the remaining *outside* bracket in (18). The net result for $m_{n \leq 2L}^{(2)}$ is

$$m_{n \leq 2L}^{(2)} = (2L + 1)!(2L)!(2L - 1)! \times c_{n \leq 2L}, \tag{19}$$

with exactly the same expression for $c_{n \leq 2L}$ as before, (17). Thus, we have shown $m_{n \leq 2L}^{(1)} = m_{n \leq 2L}^{(2)}$.

Next, consider the coefficients where $2L + 1 \leq n \leq 3L$. There are still contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ from the line (14a), as above, of the form $(2L + 1)!(2L)!(2L - 1)! \times c_n$, and these contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ still turn out to be the same. But in this case the sums contributing to c_n give

$$\sum_{j=0}^{2L-1} \sum_{k=0}^{2L-1-j} \sum_{l=0}^{2L} \delta_{l,n-k} \Big|_{2L+1 \leq n \leq 3L} = \sum_{j=0}^{2L-1-(n-2L)} \sum_{k=(n-2L)}^{2L-1-j} 1 = \frac{1}{2}(4L + 1 - n)(4L - n). \quad (20)$$

Moreover, from applying lemma 2, there are now contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ from the second line, (14b), where the respective Z s are placed to the *left* of the A s. Following steps similar to those above, it is not difficult to see that these other terms contribute the *same amount* to either $m_n^{(1)}$ or $m_n^{(2)}$, for $2L + 1 \leq n \leq 3L$. Namely, $(2L + 1)!(2L)!(2L - 1)! \times$

$$\sum_{j=0}^{2L-1} \sum_{k=0}^{2L-1-j} \sum_{l=0}^{2L} \delta_{l,n+j-4L} \Big|_{2L+1 \leq n \leq 3L} = \sum_{j=4L-n}^{2L-1} \sum_{k=0}^{2L-1-j} 1 = \frac{1}{2}(n - 2L + 1)(n - 2L). \quad (21)$$

Thus, the net result is $m_{2L+1 \leq n \leq 3L}^{(1)} = m_{2L+1 \leq n \leq 3L}^{(2)} = (2L + 1)!(2L)!(2L - 1)! \times c_{2L+1 \leq n \leq 3L}$ with

$$c_{2L+1 \leq n \leq 3L} = \frac{1}{2}(4L + 1 - n)(4L - n) + \frac{1}{2}(n - 2L + 1)(n - 2L) = 10L^2 - 6Ln + L + n^2. \quad (22)$$

Finally, consider the coefficients for $3L + 1 \leq n \leq 6L$. These are given by an elementary reflection symmetry: $m_n^{(1)} = m_{6L-n}^{(1)}$ and $m_n^{(2)} = m_{6L-n}^{(2)}$. Thus, $m_n^{(1)} = m_n^{(2)} = (2L + 1)!(2L)!(2L - 1)! \times c_{6L-n}$ for $3L + 1 \leq n \leq 6L$. \square

As a check, the coefficients must sum to give the number of generic terms that appear in three nested $(2L + 1)$ -brackets (i.e. in either $[[[\dots]\dots]\dots]$ or $[[\dots]\dots[\dots]]$). That is $\sum_{n=0}^{6L} m_n = ((2L + 1)!)^3$. Equivalently,

$$\sum_{n=0}^{6L} c_n = 2L(2L + 1)^2. \quad (23)$$

This condition is indeed satisfied by the c_n given in (11).

3. Conclusion

Perhaps N -brackets and algebras have an important role to play in physics, as originally suggested by Nambu. Recently there has been considerable interest in N -brackets, especially 3-brackets, as expressed in the physics literature (see [2] and references therein). These ideas await further development.

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